Magnetic collapse of a neutron gas: Can magnetars indeed be formed?

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Abstract. A relativistic degenerate neutron gas in equilibrium with a background of electrons and protons in a magnetic field exerts its pressure anisotropically, having a smaller value perpendicular to than along the magnetic field. For critical fields the magnetic pressure may produce the vanishing of the equatorial pressure of the neutron gas. Taking this as a model for neutron stars, the outcome could be a transverse collapse of the star. This fixes a limit to the fields to be observable in stable neutron star pulsars as a function of their density. The final structure left over after the implosion might be a mixed phase of nucleons and a meson condensate, a strange star, or a highly distorted black hole or black "cigar", but not a magnetar, if viewed as a superstrongly magnetized neutron star. However, we do not exclude the possibility of superstrong magnetic fields arising in supernova explosions which lead directly to strange stars. In other words, if any magnetars exist, they cannot be neutron stars.

1 Introduction

Gravitational collapse occurs in a body of mass M and radius R when its rest energy is of the same order as its gravitational energy, i.e., $Mc^2 \sim GM^2/R$. We would like to argue that for a macroscopic magnetized body, e.g., composed of neutrons in an external field $\vec{B}$ ($|\vec{B}| \equiv B$), new physics arises and a sort of collapse occurs when its internal energy density U is of the same order as its magnetic energy density $\vec{\mathcal{M}} \cdot \vec{B}$, where $\vec{\mathcal{M}}$ is the magnetization. This problem is interesting in the context of both cosmology and astrophysics, as for instance in the study of objects such as neutron stars. A gas of neutral particles having an anomalous magnetic moment (as a model for neutron stars (NSs); here we assume, as usually, a background of electrons and protons in β equilibrium, which is demanded by Pauli's principle to guarantee neutron stability), when placed in extremely strong magnetic fields, has a non-linear (ferromagnetic) response to the external field and is also unstable due to the vanishing of the transverse pressure for surface fields strong enough ($B_{\text{surf}} \geq 10^{16}\,\text{G}$). In this phenomenon quantum effects play an essential role due to the coupling of the particles' spins to the microscopic field $\vec{B}$ seen by the particles (spin-polarization). For fields of this order of magnitude there are values of the density for which the magnetic energy of the system becomes of the same order of magnitude as its

total energy. Under these physical conditions any structure of superdense matter composed of neutral particles having a magnetic moment may undergo a transverse collapse when its pressure perpendicular to $\vec{B}$ vanishes. This implosion is driven by the same mechanism described in [1] for charged particles.

We present in this paper, which is an elaborated version of [2], the main ideas concerning the role of ultra-strong magnetic fields in a gas of neutral particles: The standard model of NSs, as envisioned by Duncan and Thompson in their model of magnetars [3]. The fundamental result that we obtain shows that a NS, i.e., a neutron gas permeated by a superstrong magnetic field, is unstable and must collapse. This result seems to ban the possibility of the formation of magnetars. We stress in this respect the fact that similar results were obtained by two different groups: Khalilov [4] and Ghosh, Mandal and Chakrabarty [5]. Although both teams of researchers arrived at the same conclusion as we did, quite different approaches were pursued.

This paper is organized as follows: Sect. 2 reviews the concept of anisotropic pressure in self-gravitating systems like NSs. In Sect. 3 are given the tools to construct the energy-momentum tensor of a neutron gas. Section 4 discusses the main differences between the classical and quantum collapse of a gas configuration in an approximation-independent way, based on the sign of the electromagnetic

response of the medium to the external applied field. In Sect. 5 the dynamics of the neutron gas composing a NS is studied in the presence of a magnetic field, followed in Sect. 6 by the derivation of its thermodynamical potential and magnetization. Section 7 discusses the conditions for the collapse to take place, and an application of this physics to the stability analysis of the proposed magnetars is given. The main result of this paper shows that such objects should not form if they are envisioned as a standard neutron gas in Fermi beta equilibrium as is claimed in the original idea introducing the concept of a magnetar. Some closing remarks and further potential applications of this theory are part of the final section.

2 Anisotropic pressures in self-gravitating systems

The issue of local anisotropy in pressures was extensively reviewed by Herrera and Santos [6] in a general relativistic approach. These authors present several physical mechanisms for its origin in both extremely low and very high density systems, which may include astrophysical compact objects. In the case of highly dense systems, it was pointed out that "exotic" phase transitions could occur during gravitational collapse, which is exactly the problem we are concerned with in this paper. However, the more fundamental problem regarding the appearance of anisotropic pressures in strongly magnetized compact stars was left open. More recently, Mak and Harko [7] presented a class of exact solutions of Einstein's equations corresponding to anisotropic stellar configurations which can describe realistic neutron stars.

We want to provide a more detailed description of the arising of anisotropic pressures in a relativistic system like a NS, an essential point in understanding the physics behind the problem of stability of ultramagnetized compact stars. We shall give firstly general arguments to support our view, and then we concentrate on the specific calculations, in the one-loop approximation, of the thermodynamic potential of the neutron star configuration and its magnetization, the properties from which the most crucial conclusions can be drawn.

To fix ideas, we shall work in the grand canonical ensemble, and we are considering a subsystem, some region inside the star. Such a subsystem is under the influence of the magnetic field created by the rest of the system, which we name $\vec{H}$ ($|\vec{H}| \equiv H$). The response of the subsystem is to polarize itself creating a magnetization $\vec{\mathcal{M}}$ in the medium (the neutron gas) satisfying the relation $\vec{H} = \vec{B} - 4\pi\vec{\mathcal{M}}$. Obviously, inside the subsystem the microscopic field is $\vec{B} = \vec{H} + 4\pi\vec{\mathcal{M}}$, since $\vec{B}$ (also called magnetic induction) and the electric field $\vec{E}$ are the *true fields* acting on the electric charges and magnetic dipoles of elementary particles [8,9]. (Note, however, that Landau in [10] uses the notation $\vec{H}$ to denote the magnetic field in vacuum). The field $\vec{B}$, as well as $\vec{E}$, satisfies the Maxwell equations for particles in vacuum. In what follows, when we are to discuss the dynamics of the particles

in the neutron gas, we will sometimes refer to $\vec{B}$ as the *external magnetic field*, as it is usually called in quantum field theory and astrophysics. For an external distant observer, $\vec{B} = \vec{H}$ (in Gaussian units), since the magnetization is assumed to exist only inside the star. We emphasize that actually $\vec{B}$ and $\vec{H}$ are external fields within different contexts: $\vec{H}$ is external to the *subsystem* object of study in the grand canonical ensemble, whereas $\vec{B}$ is external to any particle chosen in the subsystem (it feels, in addition to $\vec{H}$, the contribution from the magnetization field $4\pi\vec{\mathcal{M}}$ due to the other particles of the subsystem).

In the case of a gas of electrically charged particles in an external constant magnetic field $\vec{B}$, in classical electrodynamics, it is the Lorentz force $\vec{F} = e\vec{v} \times \vec{B}/c$ that is the source of an asymmetry in the pressure components parallel and perpendicular to $\vec{B}$. By writing $e\vec{v} = \vec{j}\Delta V$, where $\Delta V = dx_1 dx_2 dx_3$, calling $f_i = F_i/\Delta V$ the ith component of the force density, and substituting $\vec{j} = c\nabla \times \vec{\mathcal{M}}$, one has

$$f_i = -(\partial_i \mathcal{M}_s)B_s + (\partial_s \mathcal{M}_i)B_s . \qquad (1)$$

By multiplying by $\Delta V = dx_1 dx_2 dx_3$ and assuming $B_s = B\delta_{s3}$ and $\partial\mathcal{M}_i/\partial x_3 = 0$ (actually we also have $\mathcal{M}_i = \mathcal{M}\delta_{i3}$), only the first term in (1) remains nonzero, and one recovers the expression for the Lorentz force, which is obviously perpendicular to the field $\vec{B}$. For the corresponding pressure it yields $P_{L\perp} = -\vec{\mathcal{M}} \cdot \vec{B}$. This is a classical effect and obviously $P_{L\perp}$ must be added to the usual kinetic isotropic pressure P_0, so that the total transverse pressure becomes $P_\perp = P_0 + P_{L\perp}$. As in classical electrodynamics, by the Lenz law, $\vec{\mathcal{M}}$ is opposite to $\vec{B}$ (spin effects are neglected); then $\vec{\mathcal{M}} \cdot \vec{B} < 0$, and $P_{L\perp} > 0$. The opposite case occurs when $\vec{\mathcal{M}}$ is parallel to $\vec{B}$, which occurs in the quantum case, i.e., when spin effects are taken into account. We should be alert, however, that in the definitions and derivations that follow use will be made of the magnitudes $|\vec{B}|$ and $|\vec{\mathcal{M}}| \equiv \mathcal{M}$ of the two vectors $\vec{B}$ and $\vec{\mathcal{M}}$ instead of the vectors themselves.

3 The energy-momentum tensor of a neutron gas

Based on more fundamental grounds, one may write the general structure of the energy-momentum tensor of a neutron gas in an external field $\vec{B}$ in the same way as one does to construct general tensors, like the polarization operator tensors [11], for instance. In an external field $F_{\mu\nu}$, in addition to the basic 4-velocity vectors of the medium, u_μ, and the particle momentum k_μ, we have two extra vectors $F_{\mu\nu}k_\nu$, $F^2_{\mu\nu}k_\nu$, to form a basis of independent vectors (in what follows we shall use the notation $F^2_{\mu\nu} = F_{\mu\lambda}F_{\lambda\nu}$). From them we may build a set of basic tensors which, together with the tensors $\delta_{\mu\nu}$, $F_{\mu\nu}$, $F^2_{\mu\nu}$, serves as a basis in terms of which we can expand any tensor structure related to the particle dynamics – in particular, the energy-momentum tensor. However, to get rid of tensor structures containing off-diagonal terms,

which would correspond to unwanted shearing stresses in the fluid rest frame $u = (0, 0, 0, u_4)$, we exclude some of them, i.e., $k_\mu k_\nu$, $k_\nu F_{\mu\lambda} k_\lambda$, $F_{\mu\nu}$, $u_\mu k_\nu$, or any of their combinations. By following the arguments used in [11], we conclude that we are left in the present case with three basic tensors: $\delta_{\mu\nu}$, $F_{\mu\nu}^2$, $u_\mu u_\nu$, to describe the dynamics of such a neutron gas. Thus, the structure of such an energy-momentum tensor is then expected to be of the form

$$\mathcal{T}_{\mu\nu} = a\delta_{\mu\nu} + bF_{\mu\nu}^2 + cu_\mu u_\nu, \qquad (2)$$

where $\mu, \nu = 0, 1, 2, 3$, and $a = p$ is the isotropic pressure term, $b = \mathcal{M}/B$ and $c = U + p$. In the present case the second of these tensors can be written in a simpler form as $F_{\mu\nu}^2 = -B^2 \delta_{\mu\nu}^\perp$. The tensor (2) has then the spatial eigenvalues $p - B\mathcal{M}, p - B\mathcal{M}, p$, and the time eigenvalue $-c = -U - p$, since $u_\mu u_\mu = -1$. These eigenvalues exhibit the anisotropy in the pressures perpendicular and parallel to B.

In dealing with a quantum gas in an external field we shall assume that the sources of the field $\vec{H}$ are either classical (currents) or are due to quantum effects. Although it is out of the scope of this paper to discuss the mechanism for producing such a field, we suggest, however, some viable sources able to induce a self-consistent field, as for instance a condensate of the vector meson ρ, the neutron spin–spin ferromagnetic [12] coupling (see below), or even by diquark [13] condensation.

As in the case of the electron in an atom the basic dynamics in our present case is described by the Dirac equation in an external field, in place of the Lorentz force. According to the Ehrenfest theorem, classical dynamics is contained in the adequate average of quantum dynamics. But quantum dynamics leads also to several new phenomena not having a classical partner. After solving the Dirac equation one gets the energy eigenvalue spectrum [14]. This energy depends on the microscopic magnetic field B through some interaction term in the initial Lagrangian. These energy eigenvalues, after the quantum statistical average, determine the thermodynamic properties, such as the neutron gas pressure. If the coupling constant is turned to zero, the particles would not feel any pressure coming from the external field. Thus, since the classical Maxwell stress tensor of the field H does not express by itself the interaction with the particles, i.e., it expresses the momentum and energy of the external field, we have no need to add it to the expression (4) below. The tensor (4) already contains the basic tensor structures of the problem, including the Maxwell stress tensor of the field B, which depends on the external field H and the magnetization $|\vec{\mathcal{M}}|$.

The total external field B contributes with virtual particles, expressed through the Euler–Heisenberg vacuum terms arising in the regularization of the quantum vacuum terms [15]. These vacuum terms appear in the calculation of the basic statistical quantity, the thermodynamical potential: $\Omega \equiv -\beta^{-1} \ln \mathcal{Z}$, where $\mathcal{Z}$, the grand partition function, is built up on the particle spectrum. Our thermodynamical potential is the sum of two terms, $\Omega = \Omega_{\text{st}} + \Omega_{\text{V}}$, the finite statistical term Ω_{st} plus the vac-

uum field contribution Ω_{V}, which is divergent. In the process of regularization, it absorbs the classical field energy density term $B^2/8\pi$. We observe here that Landau, on p. 69 of [16], uses the specific term *thermodynamic potential*, denoted by Φ in that reference, to denote what in western literature is known as the Gibbs free energy, which is denoted by G. Our thermodynamic potential is just what Landau defines as a "new thermodynamic potential Ω", but we have taken it per unit volume, which is dependent on T and μ in the absence of an external field, and in our case is dependent also on B. Observe that $\Omega = F - G$. In the zero field case, it would be $\Omega = -P$, where P is the isotropic pressure. Due to the spatial anisotropy introduced by the magnetic field B, the pressures are not the same in all directions, and only in the direction parallel to B it acquires the value $\Omega(B) = -P_3$ (see below).

The coupling of the spin dipole moment of neutrons in an external magnetic field B produces a loss of rotational symmetry of the particle spectrum (in what follows we will consider B along the x_3 axis). From the spectrum, which is expressed in terms of B, by the standard methods of finite temperature quantum field theory we obtain the thermodynamical potential (per unit volume) $\Omega = \Omega(B)$, and from it all the thermodynamic properties of the system, in particular its magnetization, as is done by Huang [17] (p. 237), which is the statistical average $\mathcal{M} = -(\partial\Omega/\partial B)$.

Notice that this definition is consistent with our convention by which thermodynamical quantities are defined in terms of the microscopic magnetic field B acting on the particles as an independent variable (see for instance [9]), instead of using the quantity H. Thus, the thermodynamical variable conjugate to the magnetic field B is the quantity $\mathcal{M}(B)$, the system magnetization as introduced above. We may write then

$$\Omega = -\int \mathcal{M} dB - P_0, \qquad (3)$$

where $P_0 = -\Omega(0)$ is the term corresponding to the zero magnetic field pressure. One must emphasize that in the quantum relativistic case Ω depends on B non-linearly. From the explicit expression for Ω given in another section below, one finds that the dependence of the energy spectrum on the particle momentum is not rotationally invariant. This fact determines a reduction of the symmetry of the otherwise isotropic thermodynamic properties of the system such as the pressure, which is expected to be axially symmetric for the reasons pointed out above.

By using the Green function method it is found that the energy-momentum tensor of matter in an external constant magnetic field obeys the general structure (2), and we have [1]

$$\mathcal{T}_{\mu\nu} = \left(T\partial\Omega/\partial T + \sum \mu_r \partial\Omega/\partial\mu_r\right)\delta_{4\mu}\delta_{4\nu} \qquad (4)$$
$$+ 4F_{\mu\lambda}F_{\nu\lambda}\partial\Omega/\partial F^2 - \delta_{\mu\nu}\Omega,$$

where r runs over the species involved. Below we will take $r = n, p, e$ to describe the neutron, proton and electron component of the star gas. Expression (4) in the zero field limit reproduces the usual isotropic energy-momentum

tensor $\mathcal{T}_{\mu\nu} = P\delta_{\mu\nu} - (P+U)\delta_{4\mu}\delta_{\nu4}$ of a perfect fluid. From (4) the spatial components are $\mathcal{T}_{33} = P_3 = -\Omega, \mathcal{T}_{11} = \mathcal{T}_{22} = P_\perp = -\Omega - B\mathcal{M}$. The time component $\mathcal{T}_{00} \equiv -U_r = -TS_r - \mu_r N_r - \Omega_r$, in which $U_r \sim \mu_r N_r \sim \Omega_r$ are quantities of the same order of magnitude. We shall assume that inside a stable NS there is locally a balance between the mechanical stresses (and forces) from the pressure exerted by the quantum gas and the gravitational stress, i.e., the gravitational force per unit area, exerted by the star mass, as defined below by (6) and (7). By abusing a bit of the phrasing we shall refer in this paper to this last equation as the "gravitational pressure", in an attempt to turn to more meaningful discussions regarding effects of the gas pressure and those stemming from its own gravity. In the case of a neutron star the correct treatment of this problem requires general relativistic corrections [18]. However, for the general purpose of this paper we consider here the problem in a rough manner, in a similar fashion as it was done in the white dwarf case with zero magnetic field [17], in Newtonian gravity. If $\mathrm{d}S_i$ is a surface element and $\mathrm{d}x_i$ an increase in the coordinate $i = 1, 2, 3$, one can express the balance between the work done on the star and the variation of its gravitational energy, $E_\mathrm{g} \equiv GM^2 R^{-1}$, under a small change of volume ΔV by

$$\int \left(-\mathcal{T}_{ij}\mathrm{d}S_j + \frac{\partial E_\mathrm{g}}{\partial x_i} \right) \mathrm{d}x_i = 0 , \qquad (5)$$

where the quantity in parentheses must be zero; it describes the balance of forces. We will use now cylindrical coordinates with the z axis parallel to B, and assume that the shape of the deformed star can be (very roughly) approximated by the simplified model of a cylinder of height Z and radius $r_\perp$ (we could equally well use as a simple model a cylinder of height Z ended by two half-spheres of radius $r_\perp$; then (6) and (7) below would differ by unimportant numerical factors). We assume also that the changes of the cylinder's volume ΔV and surface ΔS are small, so that the total surface of the star $S = 2\pi r_\perp(r_\perp + Z)$ is approximately constant. Then we have $\mathrm{d}S_\perp = 2\pi r_\perp \mathrm{d}z$, $\mathrm{d}S_3 = 2\pi r_\perp \mathrm{d}r_\perp$ and integrating we get the equilibrium between the gravitational and gas pressures $P_\perp = P_{\mathrm{g}\perp}$ and $P_{33} = P_{\mathrm{g}33}$, where

$$P_{\mathrm{g}\perp} = \frac{1}{2\pi r_\perp Z} \frac{\partial E_\mathrm{g}}{\partial r_\perp} \qquad (6)$$

and

$$P_{\mathrm{g}33} = \frac{1}{\pi r_\perp^2} \frac{\partial E_\mathrm{g}}{\partial z}. \qquad (7)$$

Equations (6) and (7) are to be interpreted respectively as the transverse and longitudinal "gravitational pressures" in the sense defined above. By assuming that $\frac{\partial E_\mathrm{g}}{\partial r_\perp}$ and $\frac{\partial E_\mathrm{g}}{\partial z}$ are quantities of the same order, to preserve the balance of transverse pressures when $P_\perp$ decreases (due to the increase of the product $B\mathcal{M}$), the quantity $1/2\pi r_\perp Z$ must decrease. This is achieved whenever Z increases faster than $r_\perp$ decreases (all this can be seen by assuming that the total surface $S = \mathrm{const}$ and $\mathrm{d}r_\perp/\mathrm{d}\tau < 0$, where τ is a parameter like time). In that case the outcome

is a stretching of the body along the direction of B. Thus, the anisotropy of the pressures in our problem suggests that matter in the body is also distributed anisotropically, which leads to prolate isobaric surfaces[1].

The condition $P_\perp = 0$ implies also that $U \sim B\mathcal{M}$. As $P_3 > 0$, an instability arises in the system leading to a transverse collapse. Thus, as pointed out in the introduction, new physics arises in connection with the vanishing of the transverse pressure, $P_\perp = 0$. This peculiar behavior will be discussed in the forthcoming sections.

By writing $\mathcal{M} = (B - H)/4\pi$, one may formally write $\Omega = -\frac{1}{8\pi}B^2 + \frac{1}{4\pi}\int H\mathrm{d}B - P_0$. We recall that as $\Omega \equiv F - G$, the last expression is consistent with what would be obtained in the classical non-relativistic case [19], where $F = F_0 + \int H\mathrm{d}B/4\pi$ is the Helmholtz free energy and $G = F + \int \mathcal{M}\mathrm{d}B + P_0 = G_0 + B^2/8\pi$ is the Gibbs free energy. Because of our definition of $\mathcal{M}$, the last term is given in terms of B and not in terms of H. As we have

$$\mathcal{T}_{33} = -\Omega = \frac{1}{8\pi}B^2 - \frac{1}{4\pi}\int H\mathrm{d}B + P_0, \qquad (8)$$

and also

$$\mathcal{T}_{11} = \mathcal{T}_{22} = -\Omega - B\mathcal{M} = -\frac{1}{8\pi}B^2 + \frac{1}{4\pi}\int B\mathrm{d}H + P_0, \qquad (9)$$

it is straightforward to see that the spatial components of the energy-momentum tensor $\mathcal{T}_{ij}$ $(i, j = 1, 2, 3)$ can be rewritten as

$$\mathcal{T}_{ij} = P_0\delta_{ij} - \mathcal{T}_{ij}^M(B, H) + \mathcal{S}(B)_{ij}, \qquad (10)$$

where $\mathcal{S}(B)_{ij} = \frac{1}{4\pi}\left[B_i B_j - \frac{1}{2}(B^2)\delta_{ij} \right]$ is the Maxwell stress tensor for the microscopic field B, and $\mathcal{T}_{ij}^M(B) = \frac{1}{4\pi}\left[H_i B_j - \left(\int B\mathrm{d}H \right)\delta_{ij} \right]$ is the Minkowski tensor for relativistic non-linear media (see below). This last term reduces to the usual expression in the non-relativistic limit if H depends linearly on B [9]. The definition (10) expresses the total pressure as a sum of an isotropic pure mechanical pressure (independent of the electromagnetic field) plus a pressure coming from the Minkowski tensor due to the interaction of the external field H with the microscopic field B, plus the Maxwell stress tensor of the microscopic field B. If $H = 0$, $B = 4\pi\mathcal{M}$ and $\mathcal{T}_{ij} = P_0\delta_{ij} + \mathcal{S}(B)_{ij}$. In this case the magnetic field is kept self-consistently.

At this point we want to refer to the recent paper by Khalilov [4] where a problem similar to the present one is studied. However, the expression for the stress tensor in a medium is taken as given only by the linear approximation of the Minkowski tensor term. This approach is not justified in the relativistic case. Further, the Maxwell tensor of the microscopic field B and the isotropic $P_0\delta_{ij}$ terms are omitted. This leads the author to wrongly conclude that the collapse occurs like in the classical case (see below), that is, along the $\vec{B}$ field, in contradiction to our present results and those of [2,1]. A consistent approach,

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[1] This is an effect opposite to the oblateness of the Sun, Earth and other planets due to the effective decrease of the transverse gravitational force by the centrifugal force

as we have followed here, and discussed in the accompanying paper [20], leads to opposite results compared to those of [4] in what concerns the spatial direction of collapse, although the fundamental issue regarding the collapse of the neutron star remains taking place in both theories.

Note, in addition, that [6,7], where the problem was also studied, did not take into account the dynamical effects of the strong (and superstrong) magnetic fields supposed to exist in the core of canonical NSs; see for instance [1, 22–24]. Thence, it is the contention of this paper to address this open issue. The novel results obtained here point to the occurrence of new physics and processes in the (relativistic) astrophysics of compact objects that were not manifest in previous papers.

4 Classical versus quantum collapses

In [1] we found that a relativistic degenerate electron gas placed in a strong external magnetic field B is confined to a finite set of Landau quantum states. As the field is increased the maximum Landau quantum number is decreased favoring the arising of either a paramagnetic or a ferromagnetic response through a positive magnetization $\mathcal{M}$, up to the case in which only the ground state is occupied. The gas then becomes topologically one-dimensional, and in consequence the pressure transverse to the field vanishes for fields $B = \Omega/\mathcal{M}$ [1]. Thus, the electron gas becomes unstable due to the decrease of the transverse pressure for fields strong enough, and the outcome is a collapse.

For neutrons the magnetization is always positive (see the arguments below) and non-linear, which leads to a sort of ferromagnetic behavior. For fields strong enough the pressure transverse to the field, $P_\perp = -\Omega - B\mathcal{M}$, is considerably decreased and may vanish. If we assume that extremely magnetized NSs, as the Duncan and Thompson magnetars [3], have fields $H \sim 10^{15}$ G, and that inside the star B increases by following a dipole law $B(r') = B_{\text{surf}}/r^3$, we expect near its surface magnetic fields ranging from 10^{16}–10^{17} G [3,27] up to values of order 10^{20} G in its core [23], where the field is maintained self-consistently, i.e., $H = 0$. For fields of this order of magnitude superdense matter composed of neutral particles having a magnetic moment may undergo a transverse collapse since $P_\perp$ vanishes. As discussed below, the emerging physics seems to ban the possibility of magnetar formation. The outcome of such a collapse might be a compact star endowed with a canonical magnetic field, as discussed below.

In the classical case in which the response of the medium is due to the Lenz law, the magnetization is opposite to the external field H and it may happen that $\mathcal{M} < 0$. This also occurs in the diamagnetic case. Then $H > B$ and $P_\perp > P_3$. Note that the opposite occurs in some permeable materials where $\mathcal{M} > 0$ and $H = B - 4\pi\mathcal{M}$ is small in comparison to either $\mathcal{M}$ and B; this is due to ferromagnetic effects which have a quantum origin, as in the neutron gas.

In the case of a classical magnetized gas, as $P_\perp > P_3$, this leads to the Earth-like oblatening effect described

above. But opposite to this, for the critical quantum configuration of the NS gas the coupling of the spin magnetic dipole with the magnetic field $\vec{B}$ plays the main role, and $\mathcal{M} > 0$ (see (21) below and the subsequent discussion where this is shown explicitly), leading to ferromagnetic effects. The situation then is reversed and $P_\perp$ is smaller than P_3 in the amount $B\mathcal{M}$ and it vanishes for $\frac{1}{8\pi}B^2 = \frac{1}{4\pi}\int B\mathrm{d}H + P_0$ leading, conversely, to a prolate configuration.

In classical electrodynamics [10] it is suggested that the total pressure is given by the sum of the Maxwell stress tensor $\mathcal{S}_{\mu\nu}$ plus an isotropic pressure (P_0) term. In the case of a constant magnetic field parallel to the x_3 axis, the total pressure tensor reads $\mathcal{T}_{ij} = P_0\delta_{ij} + \mathcal{S}_{ij}$ or $P_3 = P_0 - B^2/8\pi$ and $P_\perp = P_0 + B^2/8\pi$.

As pointed out before, the body deforms under the action of these anisotropic pressures. If the longitudinal pressure decreases, the body flattens along the magnetic field [21]. Thus, in this pure classical case, for the extreme limit of flattening, $P_3 = 0$ and $P_\perp = P_0 + B^2/8\pi$, and the body would collapse as a disk or a ring perpendicular to the field. Starting from general relativistic considerations has been reported [22] the existence of a maximum magnetic field for the case having stationary configurations of NSs. (We interpret that result as indicating the occurrence of a classical collapse for fields larger than the maximum.) This field induces a toroidal configuration, which is topologically equivalent to a ring. In the quantum case, for degenerate fermions, as $\mathcal{M} > 0$, it is $P_\perp = -\Omega - B\mathcal{M}$ which is decreased by increasing B. As the NS is in equilibrium under the balance of neutron and "gravitational" pressures, the body stretches along the direction of the magnetic field. Thus, for any density there are values of the field B high enough so that these pressures cannot counterbalance each other leading to a collapse perpendicular to the field for $P_\perp = 0$. This collapse would leave as a remnant nucleons plus a Bose–Einstein-like condensate, a hybrid or strange star[2] with canonical magnetic field [28], or a distorted ("cigar-like") black hole.

Our previous considerations are approximation independent. In order to discuss a specific model, we shall start by computing the free particle spectrum for neutral particles in a magnetic field.

5 The neutron gas in a magnetic field

For free neutrons in a magnetic field B we have the Dirac equation for neutral particles with an anomalous magnetic moment [14]:

$$(\gamma_\mu\partial_\mu + m + \mathrm{i}q\sigma_{\mu\lambda}F_{\lambda\mu})\psi = 0, \qquad (11)$$

where $\sigma_{\mu\lambda} = \frac{1}{2}(\gamma_\mu\gamma_\lambda - \gamma_\lambda\gamma_\mu)$ is the spin tensor, and $F_{\lambda\mu}$ is the electromagnetic field tensor describing B. By solving this equation we get the eigenvalues [14]

$$E_n(p, B, \eta) = \sqrt{p_3^2 + (\sqrt{p_\perp^2 + m_n^2} + \eta qB)^2}, \qquad (12)$$

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[2] The reader is referred to the interesting publications of [50]

where p_3, $p_\perp$ are respectively the momentum components along and perpendicular to the magnetic field B, m_n is the neutron mass, $q = 1.91 M_n$ (M_n is the nuclear magneton), $\eta = 1, -1$ are the σ_3 eigenvalues corresponding to the two orientations of the magnetic moment (parallel and antiparallel) with regard to the field B. Here we make the assumption that the magnetic moment remains constant for large magnetic fields. Actually, this is not so. Radiative corrections change its value as a function of B. This problem is, however, beyond the scope of the present approach.

From (12) we see that although the Hamiltonian is linear in B, the eigenvalues depend on B non-linearly. This makes the relativistic thermodynamic and electromagnetic properties of the system of neutrons very different from the non-relativistic case. For instance, all thermodynamic quantities, Ω, $\mathcal{M}$ (and in consequence, $H = B - 4\pi\mathcal{M}$), and N are also non-linear functions of B. The expression (12) also shows manifestly the change of spherical to axial symmetry with regard to momentum components. This *anisotropy in the dynamics* is expected to be reflected in an *anisotropy in the thermodynamic properties* of the system, as it is expressed by the difference between the transverse and longitudinal pressures discussed earlier on an approximation-independent basis.

The partition function $\mathcal{Z} = \mathrm{Tr}(\rho)$ is obtained from the density matrix describing the model $\rho = \mathrm{e}^{-\beta\int \mathrm{d}^3x(\mathcal{H}(\vec{x}) - \sum \mu_r N_r)}$ [31]. Here μ_j ($j = 1, 2, 3$) are the chemical potentials associated with the lepton, baryon and electric charge conservation, so that $\mu_n = \mu_2$, $\mu_p = \mu_2 + \mu_3$, $\mu_e = \mu_1 + \mu_3$ and $\mu_v = \mu_1$. The thermodynamical potential can be written as $\Omega = -\beta^{-1}\ln\mathcal{Z}$. Usually the eigenvalues of $\mathcal{H}$ contain the contribution from neutrons, protons, electrons and some meson species, and the densities are $N_r = -\partial\Omega/\partial\mu_r$, where $r = n, p, e, \cdots$ We call $\Omega = \sum_r \Omega_r$, $\mathcal{M} = \sum_r \mathcal{M}_r$ the total thermodynamical potential and magnetization, respectively.

A standard procedure is to work in the mean field approximation in which the meson fields σ, ρ, ω are taken as constant, as done in [23,24], through which the mass spectrum of the baryons is corrected and strong repulsive interactions between them are found. However, for simplicity we will keep the spectra in the tree approximation to obtain the one-loop approximation for Ω, and neglect the statistical contribution from meson terms in Ω as compared with those of fermions (for them $m_i\beta \sim 10^3$–10^4) except for fields $B \lesssim B_{c\rho} = m_\rho^2/e \approx 10^{20}$ G, since the contribution of the ρ vector meson condensate to $\mathcal{M}$ becomes relevant, and in analogy with $W^\pm$s [1], leads to a self-consistent spontaneous magnetization $B = 4\pi\mathcal{M} = 2\pi e N_\rho\sqrt{m_\rho^2 - eB}$, where N_ρ is the condensate density. However, for such fields the magnetic pressure, $\int \mathcal{M}\mathrm{d}B - \mathcal{M}B = -B^2/8\pi$, overwhelms the kinetic pressure term P_0 (of order 10^{36} dynes/cm^2) leading to $P_\perp < 0$, and the star is definitely unstable: it collapses. This mechanism is valid for other quasi-particle vector boson condensates, as diquarks, which may be formed in the medium even for smaller values of the field B [28,13].

6 Thermodynamical potential of a neutron gas

The Green function method [29] leads to a general expression for the relativistic thermodynamical potential. At the one-loop level, where no radiative corrections are considered, it is a generalization of the usual non-relativistic formula because of the fact that antiparticles must also be included. Particles and antiparticles contribute with chemical potentials of opposite sign, leading to sums or integrals over the quantum numbers involved, of terms containing the product of the logarithms of $(\mathrm{e}^{-(\mu_n \pm E_n)} + 1)$. In this case, when no external fields are present, a divergent term accounting for the vacuum energy appears which is usually subtracted [29]. In the presence of an external field, however, a term accounting for the vacuum contribution, must also be included: the Euler–Heisenberg energy of a vacuum in an external field [11,15,31,30]. One can obtain an expression for the thermodynamic potential of the neutron gas in the one-loop approximation: $\Omega_n = \Omega_{sn} + \Omega_{Vn}$, with

$$\Omega_{sn} = -\frac{1}{4\pi^2\beta} \sum_{\eta=1,-1} \int_0^\infty p_\perp \mathrm{d}p_\perp \mathrm{d}p_3$$
$$\times \ln\left[f^+(\mu_n, \beta) f^-(\mu_n, \beta)\right], \qquad (13)$$

where $f^\pm(\mu_n, \beta) = (1 + \mathrm{e}^{-(E_n \mp \mu_n)\beta})$ accounts, respectively, for the contribution of particles and antiparticles. The expression for the vacuum term thus reads

$$\Omega_{Vn} = \frac{1}{4\pi^2\beta} \sum_{\eta=1,-1} \int_0^\infty p_\perp \mathrm{d}p_\perp \mathrm{d}p_3 E_n, \qquad (14)$$

which is divergent. In the appendix we will show how to regularize this expression, and how to obtain the analog to the Euler–Heisenberg energy of vacuum due to the neutron contribution.

After integrating by parts in (13) its evaluation becomes easier. The Fermi distributions, which arise by differentiating $f^\pm$ with respect to p_3, are $n^\pm = 1/(1 + \mathrm{e}^{(E_n \mp \mu_n)\beta})$. In the degenerate case this expression reduces to $n^- = \theta(\mu - E_n)$ and $n^+ = 0$, since in that case only particles contribute to Ω. The resulting expression splits itself in two terms where the integrals are bounded by the Fermi surfaces $\mu - E_n(\eta = \pm 1) = 0$. These surfaces have axial symmetry, and thus, the Fermi momentum is not a definite number, given only in terms of μ_n and m_n, but on the opposite, it has infinite values. Thence, we have

$$\Omega_{sn} = -\frac{1}{4\pi^2} \sum_{\eta=1,-1} \int_0^\infty p_\perp \mathrm{d}p_\perp \frac{p_3 \mathrm{d}p_3}{E_n}$$
$$\times \theta(\mu - E_n(\eta)) . \qquad (15)$$

The θ function bound these integrals in the intervals $-p_{3F} \leq p_3 \leq p_{3F}$, where $p_{3F} = \sqrt{\mu^2 - (\sqrt{p_\perp^2 + m_n^2} + \eta y)}$ and $0 \leq p_\perp \leq \sqrt{(\mu - \eta y)^2 - m_n^2}$. After some transformations, it yields

$$\Omega_{sn} = -\Omega_0 \sum_{\eta=1,-1} \left[\frac{xf^3}{12} + \frac{(1+\eta y)(5\eta y - 3)xf}{24} \right.$$
$$\left. + \frac{(1+\eta y)^3(3-\eta y)}{24}L - \frac{\eta y x^3}{6}s \right], \qquad (16)$$

where $x = \mu_n/m_n$, $(m_n(x-1)$ is the usual Fermi energy), and $y = qB/m$. We define the functions $f \equiv f(x,\eta y) = \sqrt{x^2 - (1+\eta y)^2}$, $s \equiv s(x,\eta y) = (\pi/2 - \sin^{-1}(1+\eta y)/x)$, $L \equiv L(x,\eta y) = \ln(x + f(x,\eta y))/(1+\eta y)$. The functions $f = \langle p_F \rangle / m_n$, where $\langle p_F \rangle = \sqrt{\mu_n^2 - (m_n + \eta qB)^2}$ and we call them the average Fermi momenta for $\eta = \pm 1$. We see that $m_n - \eta qB$ behaves formally as a two-valued magnetic mass.

In the zero field limit one gets $P_0 = -\Omega_n(y = 0)$, where

$$\Omega_{sn}(y=0) = -\Omega_0 \left[\frac{xf_0^3}{12} - \frac{xf_0}{8} + \frac{1}{8}L_0 \right], \qquad (17)$$

where $f_0 = \sqrt{x^2 - 1}$ is the relative Fermi momentum p_F/m, and $L_0 = \ln(x + f_0)$. The neutron vacuum term (see the appendix) has an Euler–Heisenberg-like form:

$$\Omega_{Vn} = \frac{1}{4\pi^2} \int_0^\infty dy\, y^{-3} e^{-(m_n^2 + q^2 B^2)y}$$
$$\times [\cosh(qBmy) - 1 - (qBmy)^2/2!]$$
$$+ \frac{qB}{2\pi^2} \int_0^\infty dy\, y^{-2} \int_0^\infty dw\, e^{-[(w+m_n)^2 + q^2 B^2]y}$$
$$\times [\sinh(2qB(w+m_n)y) - (2qB(w+m_n)y)$$
$$+ (2qB(w+m_n)y)^3/3!]. \qquad (18)$$

It can be shown (see the appendix) that the more significant term in (18) is the first one, which for fields of order 10^{17} G leads to $\Omega_{Vn} \sim 10^{30}$ erg cm^{-3} and is negligibly small as compared with Ω_{sn} up to $B \sim 10^{18}$ G. Thus, we neglect it in a first approximation in what follows. We must point out, however, that since neutrons have a quark structure, a more fundamental quantity would be the vacuum quark contribution, whose order of magnitude is expected to be near Ω_{Vn}. Apart from this note, it should be emphasized that the role of the vacuum cannot be ignored for fields greater than 10^{18} G.

From $N_n = \partial\Omega/\partial\mu_n$ one gets

$$N_n = N_0 \sum_{\eta=1,-1} \left[\frac{f^3}{3} + \frac{\eta y(1+\eta y)f}{2} - \frac{\eta y x^2}{2}s \right]. \qquad (19)$$

In the limit $B = 0$ (19) reproduces the usual density of a relativistic Fermi gas at zero temperature, $N_n = N_0 f_0^3/3$.

Having an equation relating the chemical potentials, and demanding conservation of both baryonic number $N_n + N_p = N_B$ and electric charge $N_p + N_e = 0$, in principle one may solve exactly the problem in terms of the external field as a parameter. Nonetheless, we shall focus our discussion on the properties of the equation of state. Note in passing that our expressions for the spectra and densities of neutrons and protons are similar to those of [32] of a neutron gas in a magnetic field, but we get different equations of state.

Finally, for the magnetization, given as $\mathcal{M}_n = -\partial\Omega_n/\partial B$, we have

$$\mathcal{M}_n = -\mathcal{M}_0 \sum_{\eta=1,-1} \eta \left[\frac{(1-2\eta y)xf}{6} \right. \qquad (20)$$
$$\left. - \frac{(1+\eta y)^2(1-\eta y/2)}{3}L + \frac{x^3}{6}s \right],$$

where $N_0 = m_n^3/4\pi^2 \sim 2.04 \times 10^{39}$, $\Omega_0 = N_0 m_n \sim 3.0 \times 10^{36}$, and $\mathcal{M}_0 = N_0 q \sim 2.92 \times 10^{16}$ and one can write $\mathcal{M}_n = \mathcal{M}_n^+(\eta = -1) - \mathcal{M}_n^+(\eta = +1)$, and obviously, $\mathcal{M}_n \geq 0$.

We confirmed by explicit calculation that $\mathcal{M}$ is a nonlinear function of B and, in this sense, the magnetic response is ferromagnetic. A fully ferromagnetic response demands the inclusion also of the spin–spin coupling contribution. We briefly discuss this point below.

To see why the magnetization is always positive for the neutron gas note that the magnetic susceptibility $\chi = \partial\mathcal{M}_n/\partial B$ can easily be obtained as

$$\chi = \frac{q\mathcal{M}_0}{2m_n} \sum_{\eta=\pm1} \left[xf + (1+\eta y)^2 L \right], \qquad (21)$$

which for $x > 1$, $y \leq 1$, and f real and positive, is $\chi > 0$. This means that $\mathcal{M}_n$ is an increasing function of B (or y) under such conditions. As $\mathcal{M}_n(y = 0) = 0$ and $\mathcal{M}_n(y = 1) = 2\mathcal{M}_0(1 + \pi) > 0$, this means that $\mathcal{M}_n > 0$ in the region between these two points, which is the one of interest for us (the region I discussed below and shown in Fig. 2).

The fact that we are summing over the magnetic moments oriented parallel ($\eta = -1$) and antiparallel ($\eta = +1$) to B is similar to the well-known Pauli paramagnetism in non-relativistic quantum statistics. We may consider each term $\eta = \pm 1$ as representing a phase of the system. One can draw in the x, y plane, for both $x > 0$, $y > 0$, two regions limited by the lines $x - y = \pm 1$ and $x + y = 1$ (see Fig. 1). The region I is bounded above by the line $x = y + 1$, on which the contribution from $f(\eta = +1)$, and all other terms containing $\eta = +1$ vanish: all magnetic moments are aligned parallel to B. Below, such a region is limited by $y = 0$. For points in the region $x > y + 1$ and $y > 0$ both terms containing $\eta = -1, +1$ are non-zero and real. Thus, the quantities Ω, N_n, and $\mathcal{M}$ are real and although most neutrons have their magnetic moments along B, there are some having their magnetic moments antiparallel to B. In this region, near the line $x = y + 1$, is located the curve $P_\perp = 0$, which we will discuss below.

In the region II limited by $x \geq 1 - y$, $y + 1 > x \geq y - 1$ only the term $\eta = -1$ is real and the term $\eta = +1$ becomes purely imaginary, and all neutrons would have their magnetic moments parallel to B. The term $\eta = -1$ contributes to Ω, N_n, and $\mathcal{M}$. For $x < y - 1$, there are no physical solutions. For $y = 0$, the magnetic field, and
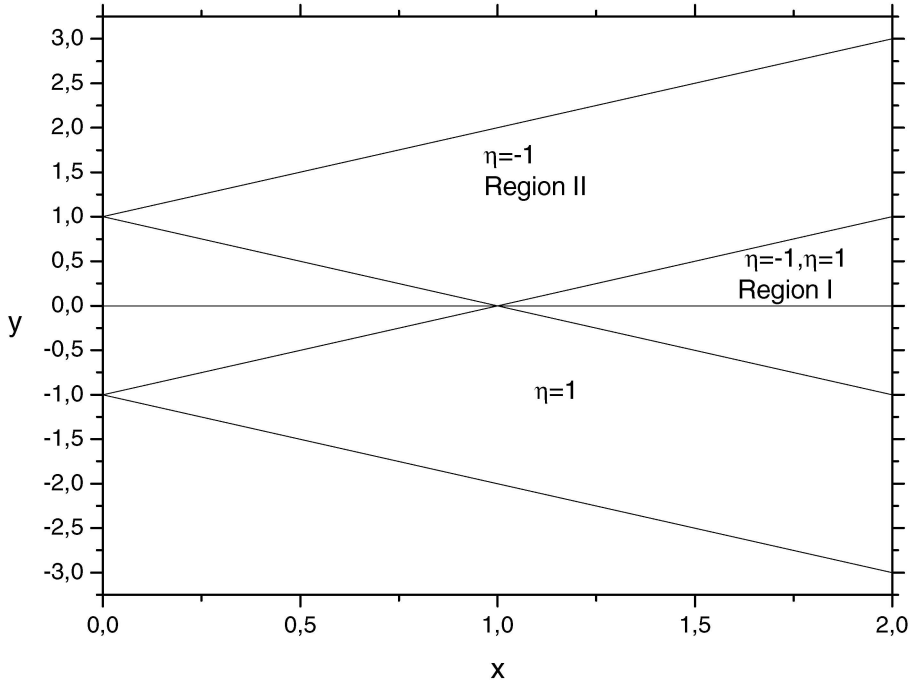
Fig. 1. This plot shows the regions in the x, y plane where the neutron magnetic moments are oriented parallel or antiparallel to the magnetic field $\vec{B}$. Special attention should be given to regions I and II, where the solutions discussed in the text are valid
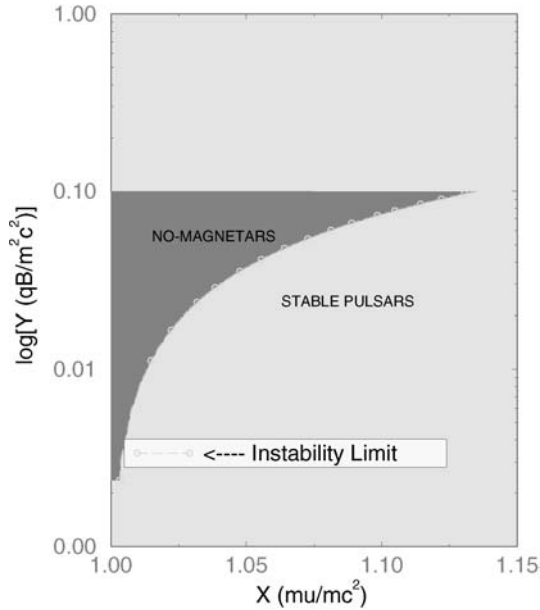

Fig. 2. The instability condition, $P_\perp = 0$, in terms of the variables x, y. A nascent strongly magnetized neutron star having a configuration such that its dynamical stage would be represented by a point above the central curve in this plot would be unstable to transversal collapse, since $P_\perp \le 0$ there

show the instability which arises from the vanishing of the transverse pressure. For densities and magnetic fields above these critical values, the stability of the electron gas is doubtful. Recently Ghosh, Mandal and Chakrabarty [5] have pointed out strong arguments against the equilibrium of an electron gas under β decay in strongly magnetized neutron stars. The situation may be modified if radiative corrections are taken into account: if $B_{ce} = m_e^2/e \sim 10^{13}$ G is the usual QED critical field, for larger fields of order $B'_{ce} \sim 4\pi/\alpha B_{ce} \sim 10^{16}$ G the contribution of the electron anomalous magnetic moment becomes significant and the problem cannot be satisfactory treated at the tree level. However, starting from the results of [1,5], one concludes that the electron gas is hardly in equilibrium for fields beyond B_{ce}. One possibility is the bosonization of the electron system, as has been recently suggested [33]. This may be accomplished through the increase of the spin–spin interaction, leading to the formation of parallel spin electron pairs, equivalent to charged vector bosons.

If we include both the normal and the anomalous magnetic moments for protons, one can give a formula for their spectrum in the external field B [14]:

$$E_p = \sqrt{p_3^2 + (\sqrt{2eBn + m_p^2} + \eta q_p B)^2}, \qquad (22)$$

We have $q_p = 2.79 M_n$. For neutrons, the critical field at which the coupling energy of its magnetic moment equals the rest energy is $B_{cn} = 1.57 \times 10^{20}$ G. For protons it is $B_{cp} = 2.29 \times 10^{20}$ G. By defining $x_p = \mu_p/m_p$, $y_p = q_p/m_p$, $b = 2e/m_p^2$, then $y_p = 2.79e/2m_p^2$. We mention also $g \equiv g(x_p, B, n) = \sqrt{x_p^2 - h(B, n)^2}$ and $h \equiv h(B, n) = (\sqrt{bBn + 1} + \eta y_p B)$. Thus for the proton thermodynamical potential we get

in consequence, the magnetization $\mathcal{M}$ vanish. For $y < 0$ (magnetic field along the negative y axis) we have the situation reverse to the one described above (see Fig. 1).

With regard to the background of electrons and protons, we recall that the case of the electron gas was discussed in [1]. We expect that for magnetic fields of order $B_{ce} = m_e^2/e \sim 10^{13}$ G and densities around 10^{30} cm^{-3} all electrons are in the Landau ground state, and the system

$$\Omega_p = -\frac{eBm_p^2}{4\pi^2} \sum_n \sum_{\pm\eta} \left[x_p g - h^2 \ln(x_p + g)/h \right], \quad (23)$$

and for its density

$$N_p = \frac{eBm_p}{2\pi^2} \sum_n \sum_{\pm\eta} g(x_p, B, n) , \qquad (24)$$

while the magnetization is given by

$$\mathcal{M}_p = \frac{em_p^2}{4\pi^2} \sum_n \sum_{\pm\eta} \Big\{ x_p g - \Big[h^2 + (\eta y_p \qquad (25)$$

$$+ (bn/2\sqrt{bBn+1})) \Big] \times \ln(x_p + g)/h \Big\},$$

where the coefficients of these formulae are $N_{0p} = em_pB/2\pi^2 \sim 4.06 \times 10^{19} B$, $\Omega_{0p} = N_{0p}m_pB \sim 6.1 \times 10^{16}B$, and $\mathcal{M}_{0p} = N_{0p}m_p = \Omega_0/B$. The maximum occupied Landau quantum number n may be given as $n_{\max} = (x_p - \eta y_p B)^2 - 1/bB$. For $B \ll B_{cp}$, so that $y_p B \ll 1$, and $x_p \geq 1$, one can take approximately $n_{\max} \sim (x_p^2 - 1)/bB$, and for fields large enough $n_{\max} = 0$. We expect that from the equation $\mu_n = \mu_p + \mu_e$, $x_p \sim x_n$, the previous expression for the proton density N_p decreases with increasing B, favoring the inverse beta decay. For fields $B \sim m_p/q_p$ and $x_p \gg 1$, $n_{\max} \geq 1$, and thus large Landau numbers are again occupied. However, for $x_n, x_p \gtrsim 1$, being quantities of the same order of magnitude, from the comparison of $\Omega_{0p}, N_{0p}, \mathcal{M}_{0p}$ with $\Omega_0, N_0, \mathcal{M}_0$ we conclude that for fields below 10^{19} G, the dominant longitudinal pressure, density and magnetization comes from the neutron gas.

7 Condition for zero transverse pressure and collapse

In the electron gas case [1], the vanishing of the transverse momentum can be guessed from the spectrum when the whole system is confined to the Landau ground state. The spectrum corresponds to a purely one-dimensional system moving along the external field, and the transverse Fermi momentum is zero. In a similar way, in the neutron gas case we observe that the threshold of zero transverse pressure $P_\perp = 0$ can be figured out from the spectrum described by (12), since the contribution from the $\eta = -1$ term is dominant (observe that the term with $\eta = +1$ contributes with a negative term to $\mathcal{M}$). We shall consider on the Fermi surface for $\eta = -1$ the quantity

$$p_{\mathrm{F}\perp\mathrm{eff}}^2 = \mu_n^2 - p_{\mathrm{F}3}^2 - m_n^2$$

$$= \left(\sqrt{p_{\mathrm{F}\perp}^2 + m_n^2} - qB \right)^2 - m_n^2, \qquad (26)$$

which we call the effective squared Fermi transverse momentum. If $B \ll 10^{20}$ G, then $q^2 B^2 \ll 2eBm_n$, and the vanishing of $p_{\mathrm{F}\perp\mathrm{eff}}$ is guaranteed if

$$p_{\mathrm{F}\perp} \sim \sqrt{2qBm_n}. \qquad (27)$$

The resulting Fermi surface would be equivalent to that for one-dimensional motion, parallel to B, i.e., for particles having energy $E_n \simeq \sqrt{p_3^2 + m_n^2}$, and in consequence the transverse momentum (and pressure) vanishes. Notice from (27) that for $p_{\mathrm{F}\perp}/m_n \sim 10^{-1.5}$ one has $qB/m_n = y \sim 10^{-3}$, which means fields of order 10^{17} G. A more accurate result is obtained, however, from the condition $\mathcal{T}_\perp = 0$.

In Fig. 2 we have drawn the equation $P_{\perp n} = -\Omega - B\mathcal{M} = 0$ in terms of the variables x, y. We observe that there is a continuous range of values of x, starting from $x = 1.005$ to $x = 1.125$ and from $y = 0.001$ to $y = 0.1$, for which the collapse takes place. The latter values of y means fields in the interval $B \sim 10^{17}$ to 10^{19} G. To these ranges of x, y corresponds a continuum range of densities, from $10^{-2}N_0$ (10^{12} g/cm^3) onwards. The transverse compression of the whole mass of the star due to flux conservation leads to an increase of B and the mechanism of collapse is enhanced.

7.1 The spin–spin coupling contribution

Although we previously used the term "ferromagnetic" to denote the non-linear response of the medium to the external field, what we have considered actually in our previous calculations is the occurrence of relativistic Pauli paramagnetism. We have not fully considered the spin–spin coupling, which would lead more definitely to Heisenberg ferromagnetism, and it is physically reasonable to expect its appearance in nuclear matter for densities high enough. This problem has been studied in [34] via the interaction through axial vector and tensor exchange channels. These authors show that if the interactions are strong enough and differ in sign, the system loses the spherical symmetry due to a mechanism independent of the one discussed in the present paper. In the case of a quark liquid [35] the problem has been studied under one gluon exchange interaction, and conditions for ferromagnetism may arise.

Although the problem requires further research, one expects that if Heisenberg ferromagnetism is to appear, it would increase the magnetization to $\mathcal{M}_s = \kappa \mathcal{M}$, where κ is the internal field parameter. If $\kappa >> 1$ our previous estimates for the vanishing of the transverse pressure might be largely exceeded, with the arising of a new spontaneous magnetization $\mathcal{M}_s \sim \kappa \mathcal{M} = B/4\pi$. This would mean that the magnetic field B could be kept self-consistently and our previous calculations would be a lower bound of the fully ferromagnetic case. The vanishing of $P_\perp$ is expected to occur surely at values of B smaller than those depicted in Figs. 2 and 3. As a rough estimation, if we assume the exchange interaction J among neutrons to be of the order of a few hundreds of MeV, and the number of nearest neighbors to be $z \sim 10$, by dividing their product by the dipole interaction energy, say, for the core of the star where $N = N_0$, we get $\kappa \sim 10^4$. This means $B \sim 4\pi\mathcal{M}_s \sim 10^{20}$ G. For such extremely large fields the magnetic coupling of quarks with B would become of the order of their binding energy through the color field producing a deconfinement phase transition leading to a

quark (q)-star, a pressure-induced transition to *uds*-quark matter via *ud*-quark condensates, as discussed in [43,44]. But fields of that order lead surely to the collapse of the star and even to the instability of the vacuum.

8 Magnetar formation and stability

Next we briefly review the basic ideas supporting the theory of magnetars and then show why these hypothetical objects cannot survive after reaching the claimed superstrong magnetic fields. We then present prospectives for a hybrid or strange star to appear as a remnant of the quantum magnetic collapse of a NS. According to Duncan and Thompson [3], NSs with very high dipole surface magnetic field strength, $B \sim [10^{14}$–$10^{15}]$ G, may form when the (classical) conditions for a helical dynamo action are efficiently met during the seconds following the core collapse and bounce in a supernova (SN) explosion [3]. A newly born NS may undergo vigorous convection during the first 30 s following its formation [39]. If the NS spins (differentially) sufficiently fast ($P \sim 1$ ms) the conditions are created for the $\alpha - \Omega$ dynamo action to be built, which may survive depletion due to turbulent diffusion. Collapse theory, on the other hand, shows that some presupernova stellar cores could acquire enough spin so as to rotate near their Keplerian equatorial velocity, the breakup spin frequency: $\Omega_{\mathrm{K}} \geq \left(\left[\frac{2}{3} \right]^3 G_N M / R^3 \right)^{1/2}$, which implies a period $P_{\mathrm{K}} \sim 0.6$ s, after core bounce. Under these conditions, fields as large as [42,3]

$$B \sim 10^{17} \left(\frac{P}{1 \, \mathrm{ms}} \right) \, \mathrm{G} \qquad (28)$$

may be generated as long as the differential rotation is dragged out by the growing magnetic stresses. For this process to efficiently operate the ratio between the spin rate (P) and the convection overturn time scale (τ_{con}), the Rossby number (R_0), should be $R_0 \leq 1$. Duncan and Thompson warned that $R_0 \gg 1$ should induce less effective mean-dynamos [3]. In this case, an ordinary dipole $B_D \sim [10^{12}$–$10^{13}]$ G may be built by incoherent superposition of small dipoles of characteristic size $\lambda \sim [\frac{1}{3}$–$1]$ km, and a saturation strength $B_{\mathrm{sat}} = (4\pi\rho)^{1/2} \lambda / \tau_{\mathrm{con}} \simeq 10^{16}$ G may be reached at the surface during this early evolution of the nascent neutron star. At such fields, the huge rotational energy of a NS spinning at $\omega_{\mathrm{NS}} \geq 1$ kHz, is leaked out via *magnetic braking* and an enormous energy is injected into the SN remnant. This process may explain the power emitted by a *plerion*.

As shown above, at the end of the SN core collapse we are left with a rapidly rotating NS endowed with an extremely strong magnetic field (ESMF) strength and a large matter density $\rho \sim [10^{14}$–$10^{12}]$ g cm^{-3}. As illustrated in Fig. 1, those are the conditions for the quantum instability to start to dominate the dynamics of the young pulsar. At this stage, the magnetic pressure inwards may pass the star energy density at its equator and the collapse becomes unavoidable. As the collapse proceeds, higher and higher

densities are reached till the point the supranuclear density may reverse the direction of the implosion. A hybrid or strange star (SS) may have formed. We explore next this plausible outcome, among the other possibilities quoted above. From that moment, the sound wave generated at the core bounce builds itself into a shock wave traveling through the star at $V_{\mathrm{SW}} \sim c/\sqrt{3}$ km s^{-1}. Although the ESMF strength could be quite large as long as the collapse advances, the huge kinetic energy, $E \sim 10^{51-52}$ erg, the mean energy obtained in calculations of energy release in neutron star phase transitions to strange (*twin*) stars [36–38] and some *prompt shock* supernovae ($E_{\mathrm{p-s}} \sim 10^{51}$ erg) [39], carried away by the shock wave drives a kind of supernova explosion inasmuch as in the quark nova model and similar scenarios [38]. Such a huge *ram* pressure may counterbalance the magnetic pressure, and even surpass it, i.e.,

$$\rho_{\mathrm{eject}} V_{\mathrm{SW}}^2 \geq \frac{B^2}{8\pi\mu_0} \left(\frac{R}{r_A} \right)^6, \qquad (29)$$

at a location from the star center equivalent to the Alfvén radius of the magnetar

$$r_A = \left(\frac{2\pi^2}{G\mu_0^2} \right)^{1/7} \left[\frac{B^4 R^{12}}{M \dot{M}^2} \right]^{1/7} \sim 80 \, \mathrm{km}. \qquad (30)$$

This radius is quite large, about 7 times the NS radius (see Table 1). Here $\dot{M}$ defines the accretion rate of the freefalling overlaying material making up the NS crust, which is left out when the transition occurs (see definitions and further details in [38], and references therein). Therefore, it is quite legitimate to expect that most of the magnetic energy stored inside the magnetosphere to be drained out of the Alfvén radius. Notice in addition that the strange star radius scales as $R_{\mathrm{SS}} = R_{\mathrm{NS}} \left(\frac{\rho_{\mathrm{NS}}}{\rho_{\mathrm{SS}}} \right)^{1/3}$, where the densities ratio reads $\frac{\rho_{\mathrm{NS}}}{\rho_{\mathrm{SS}}} \simeq 0.1$–$0.2$. Other relations between both stars can be obtained by using conservation laws or appropriate scalings.

Then the ESMF lines are pushed out and finally broken, in a process inverse to the standard accretion one, from r_A onwards into the SN remnant surroundings, as a violent explosion that dissipates a large part of the magnetic flux ($\Phi \sim B^2 r_A^2$) and the energy trapped inside the magnetar magnetosphere [28]. Energy from the magnetic field can be dissipated via vacuum polarization and electron–positron pair creation, as well as acceleration of charged matter flowing away (synchroton and curvature losses [40,41]) with the explosion and material trapped in the star magnetosphere and placed near the Alfvén radius. This is analogous to the mechanism operating during a solar flare or a coronal mass-ejection, where the very high B in a given Sun-spot is drastically diminished after flaring for a short period of time (see also [42]). In the Sun-spots outbursts, and coronal mass-ejections, launch into space, part of the Solar wind of charged and neutral particles that passes by the Earth [46,47]. In the case of an imploding NS, the phase of open magnetic field lines over which the strange star is acting as a *propeller* lasts for about $\Delta T_{\mathrm{prop}} \sim E_{\mathrm{spin}} / L_{\mathrm{prop}} \sim [10^2$–$10^3]$ s, with $L_{\mathrm{prop}} \sim 2\dot{M}c^2$
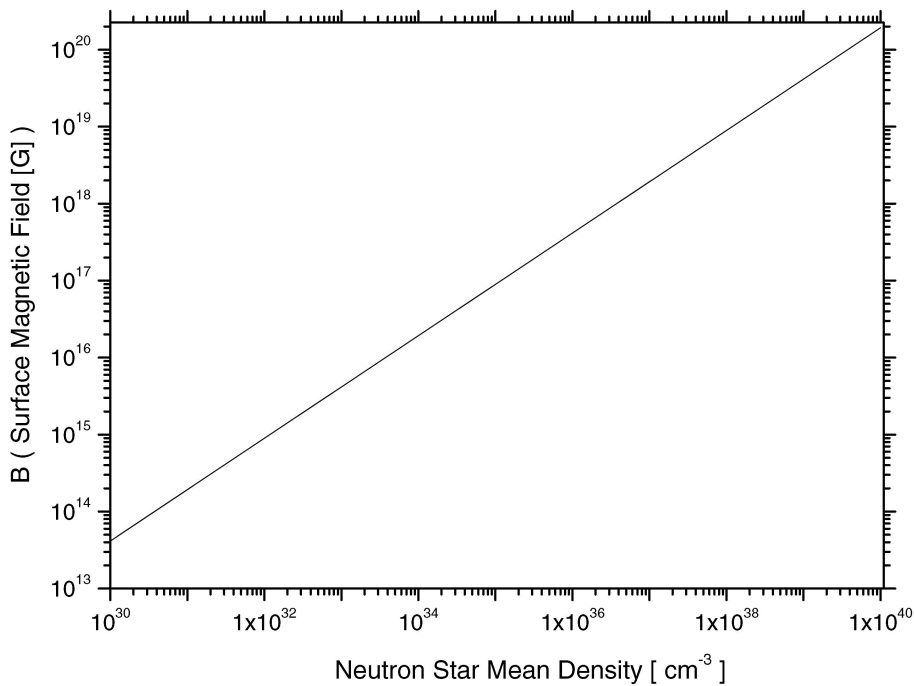
Fig. 3. The curve $P_\perp = 0$ in terms of the neutron star mean density N and the surface magnetic field B

Table 1. Parameters used in modeling the phase transition of a neutron to a strange star, as discussed in the text

	Mass M [M$_\odot$]	Radius R [km]	Period P [ms]	Magn. field B [G]	Accretion rate [M$_\odot$ s^{-1}]
Neutron star	1.5	12.5	2.0	2×10^{15}	10^{-15}
Strange star	1.5	9.5-10.0	0.5	2×10^{13}	10^{-5}

the propeller luminosity, and $E_{\rm spin} \sim I\omega_{\rm SS}^2$ the star rotational energy. Thence, the large amount of matter ejected from the strange star at such large velocities, and the pairs created in the vacuum breakdown, drains out the dipole field of the remnant below the quantum electrodynamic limit of $B_{ce} \sim 4.4 \times 10^{13}$ G [28].

To give an insight into this piece of the physics of the problem, notice that once the propeller phase is over and no more luminosity is coming from that mechanism, the magnetic field lines can recombine again if the energy released in this new stable phase is essentially the strange star rotational dipole luminosity (as the one from a millisecond pulsar), which then becomes the star's dominant mechanism of energy emission, that is,

$$4\frac{B_{\rm SS}^2 R_{\rm SS}^6 \omega_{\rm SS}^4}{9c^3} = I\omega_{\rm SS}\dot\omega_{\rm SS} . \tag{31}$$

For the parameters shown in Table 1 and the observed luminosity from fast rotating pulsars, this relation implies a new equilibrium magnetic field $B \sim 10^{12-13}$ G, which is below the quantum electrodynamics threshold.

Since all the differential rotation has been dragged up to build up the former ESMF, then nothing else remains to make the magnetic field grow to its pre-collapse value. Thence no such ultrahigh B should reappear. We may

be left with a sub-millisecond strange star [43] or a hybrid star [44] with "canonical" field strength, but not any magnetar. We note in passing that the above theoretical result is attained on the standardized assumption that the structure of the magnetic field of the pulsar is dipolar. This premise is underlying the claim by Kouveliotou et al. [45] that a magnetar had been discovered in the soft gamma-ray repeater source SGR 1806-20. Notwithstanding, for other NS (multipolar or uniform) field configurations we do expect the overall behavior here discussed to persist, since once the spin–spin coupling is taken into account the unavoidable consequence is the appearance of a ferromagnetic (axial) configuration or structure which is dipolar in nature, and therefore the theory propounded here still holds, because the magnetic field could then be amplified by a factor 10^4), putting the nascent pulsar above the threshold for stability, and the collapse ensues. Rephrasing this, one can think of this theory as a field-configuration independent constraint on initial NS magnetic field strengths.

In looking for other contexts far from those involving compact remnant stars, we noticed that recent "Tabletop Astrophysics" experiments performed by Wieman et al. [48] have succeeded in refining sophisticated techniques to switch atoms in a Bose–Einstein condensate (BEC) [48]

from states of implosion to states of re-expansion (or explosion). In certain cases they observed that some collapses appear to be rather similar to microscopic supernovae. The initial implosion is followed by an explosion in which atoms are ejected as a hollow ball or in narrow jets, like in the collapse and rebound of a exploding star that forms characteristic expanding balls or streams of outflowing gas. The Wieman team named the phenomenon "bosenovae" because of the similarities with typical supernovae. In fact, in both phenomena some material is left after the implosion as a compact object. Because of the similarity of the physics of Bose–Einstein condensates with the one we presented above, we are confident that to this our theory can also be adapted to explain these impressive results by Wieman et al. [48] and Ketterle and Anglin [26]. The main feature of these oscillating Bose–Einstein condensates, said to be a scaled-down version of either a neutron star or a white dwarf [48], is that for some critical fields what appears is an attractive force between the atoms, and the condensate implodes and rebounds driven by some sort of internal negative pressure. We claim in this paper (a detailed description of BECs phenomenology is to be given in a work in preparation [49]) that such a negative pressure could be explained in the context of the theory introduced in this paper, since a precise relationship between gas density and magnetic field strength in the BEC is settled by switching the atoms between attractive and non-attractive states. Thence the negative pressure acts as the equivalent of an attractive force among the atoms directed towards the magnetic field axis, leading to the BEC implosion.

9 Conclusions

We conclude by claiming that if a degenerate neutron gas is under the action of a superstrong magnetic field $B_{ce} \lesssim B \lesssim B_{cn}$, for values of the density typical of NS matter its transverse pressure vanishes, the outcome being a transverse collapse. This phenomenon establishes a strong bound on the magnetic field strength expected to be found in any stable neutron star pulsar, regardless of its initial field configuration, and suggests a possible endpoint in the early evolution of highly magnetized neutron stars. They could likely be a mixed phase of nucleons and a $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \sigma, \rho^\pm, \omega$ meson condensate, a hybrid or strange star, or a distorted black hole but not any magnetar at all. We point out, nonetheless, that if by any mechanism a strange star could be formed directly in a supernova explosion (which is uncertain, but not ruled out) and if a vigorous dynamo action operates in the strange quark matter bulk, "pulsars" with fields higher than B_{ce} could still be formed. In other words, if any magnetars exist, they cannot be neutron stars.

A Appendix

We use the integral representation

$$a^{1/2} = \pi^{-1/2} \int_0^\infty \frac{\mathrm{d}x}{x^2} (\mathrm{e}^{-ax^2} - 1)$$

$$= \pi^{-1/2} \int_0^\infty \mathrm{d}y y^{-3/2} (\mathrm{e}^{-ay} - 1). \tag{32}$$

We regularize the divergent term dependent on a in (32), by introducing a small quantity ϵ as the lower limit in the integral, and neglecting the term independent of a

$$a^{1/2}(\epsilon) = \pi^{-1/2} \int_\epsilon^\infty \mathrm{d}y y^{-3/2} \mathrm{e}^{-ay}. \tag{33}$$

By taking $a(\epsilon, \eta) = p_3^2 + (\sqrt{p_\perp^2 + m_n^2} + \eta qB)^2$ and substituting in (15) and performing the Gaussian integral on p_3, one obtains

$$\Omega_{\mathrm{V}n}(\epsilon) = \frac{1}{4\pi^2} \sum_{\eta=\pm 1} \int_0^\infty p_\perp \mathrm{d}p_\perp \int_\epsilon^\infty \mathrm{d}y y^{-5/2}$$
$$\times \mathrm{e}^{-\left(\sqrt{p_\perp^2 + m_n^2} + \eta qB\right)^2 y}. \tag{34}$$

By substituting $z = \sqrt{p_\perp^2 + m_n^2} + \eta qB$ one is left with the expression

$$\Omega_{\mathrm{V}n}(\epsilon) = \frac{1}{4\pi^2} \int_\epsilon^\infty \mathrm{d}y y^{-3} \mathrm{e}^{-(m_n^2 + q^2 B^2)y} \cosh(qBmy)$$
$$+ \sum_{\eta=\pm 1} \frac{\eta qB}{4\pi^2} \int_\epsilon^\infty \mathrm{d}y y^{-2} \int_{m_n + \eta qB}^\infty \mathrm{e}^{-z^2} y \mathrm{d}z. \tag{35}$$

By introducing the new variable $w = z - m_n - \eta qB$, the second integral in (35) becomes

$$\frac{\eta qB}{4\pi^2} \int_\epsilon^\infty \mathrm{d}y y^{-2} \int_0^\infty \mathrm{e}^{-[(w+m_n)^2 + q^2 B^2]y}$$
$$\times \sinh(2qB(w + m_n)y). \tag{36}$$

After subtracting from $\cosh(qBmy)$ and $\sinh[2qB(w + m_n)y]$ the first two terms in their series expansion, one can take $\epsilon \to 0$ and obtain the finite expression (18). This process is equivalent to the subtraction of divergent terms, one of which is proportional to B^2 and absorbs the classical field energy term $B^2/8\pi$.

It is not difficult to check that for fields $B \ll 10^{20}\,\mathrm{G}$, the first term in (18) is the dominant one. Its first contribution after the series expansion of $\cosh(qBmy)$ is $q^3 B^3 m_n^3 / 2\pi^2 (m_n^2 + q^2 B^2)$. For fields of order $10^{17}\,\mathrm{G}$ such a term is of order $10^{30}\,\mathrm{ergs/cm^3}$, much smaller than Ω_{sn}. But for fields near $10^{20}\,\mathrm{G}$ its contribution is comparable to that of Ω_{sn}.

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